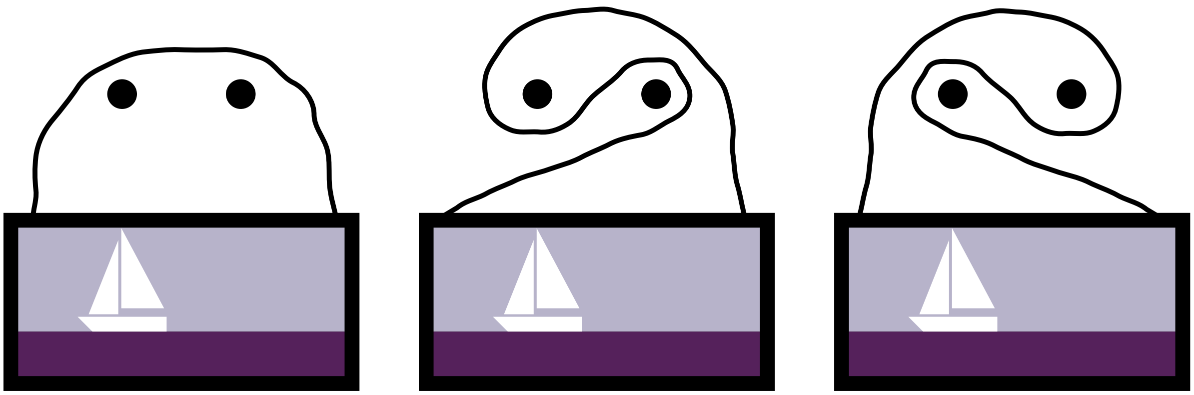
There are three exercises labeled A–C.

Here's the classic puzzle about hanging a painting. You are helping your sibling move into a new apartment. They have nailed two nails into a wall. They hand you a framed painting of a sailboat and a spool of thin cable, which can attach to the painting in two places near its top corners. They ask you to cut some cable, attach it to the painting, and hang the painting over both nails. Why both nails? For safety: If one nail falls out of the wall for some reason, then the other nail will still prevent the painting from crashing to the floor. They expect you to hang the painting as in the diagram below left.



However, you decide to play a trick on your sibling. You realize that if you hang the painting as in the diagram above center, then it is "over both nails" as requested, but it will fall to the floor if the left nail falls out. (It does not fall to the floor if only the right nail falls out.) Symmetrically, if you hang the painting as in the diagram above right, then it falls if the right nail falls out. So you wonder: Is there a way to wind the cable over both nails, so that the painting falls to the floor if *either* of the nails falls out?

A. Solve this puzzle using your knowledge of the fundamental group of $\mathbb{R}^2$ with two punctures, the fundamental group $\mathbb{R}^2$ with one puncture, and the relationship between the two. Draw a diagram, like the ones above, of the simplest solution that you can find, to convince yourself that it actually works. Do not submit just the diagram; also submit your explanation of how fundamental groups lead to that solution.

We have computed the fundamental group of $\mathbb{RP}^2$ in two ways: one based on the covering map $\mathbb{S}^2 \to \mathbb{RP}^2$ and another based on the Seifert-van Kampen theorem. Let X be the m-fold dunce cap, which equals $\mathbb{RP}^2$ when $m = 2$. We have computed $\pi_1(X)$ based on the Seifert-van Kampen theorem.

B. Is there also a computation of $\pi_1(X)$ based on a covering map $\mathbb{S}^2 \to X$? If so, then sketch the proof. If not, then explain why not. In both cases, a rigorous proof is not expected.

[This is a true story.] I draw a square with labeling (a, b, a, b). This labeled polygonal region glues to make $X = \mathbb{RP}^2$. So $\pi_1(X)$ is isomorphic to the free group on two generators a and b modulo the smallest normal subgroup containing (a, b, a, b). But I already know that $\pi_1(\mathbb{RP}^2)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. So I spend a while doing pure algebra, trying to prove that $\pi_1(X)$ (as a quotient of a free group) is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. But I'm having trouble. I'm worried that it's not even true.

C. What's your advice? Am I doing something wrong? Do I just need to keep trying?