A. [This is Section 18 Exercise 10.] Let $U \subseteq Y \times W$ be open. By the definition of the product topology, U is of the form $U = \bigcup_{i \in I} U_i \times V_i$, where $U_i \subseteq Y$ and $V_i \subseteq W$ are open. Then

$$h^{-1}(U) = h^{-1}\left(\bigcup_{i \in I} U_i \times V_i\right) = \bigcup_{i \in I} h^{-1}\left(U_i \times V_i\right) = \bigcup_{i \in I} f^{-1}(U_i) \times g^{-1}(V_i).$$

But those inverse image sets are open in X and Z respectively, because f and g are continuous. So $h^{-1}(U)$ is a union of open sets in the product topology on $X \times Z$. So $h^{-1}(U)$ is open. This argument shows that h is continuous.

B. For all $x \neq y \in X$, define $d(x, y) = 1$. Of course, $d(x, x) = 0$ for all x. This d is a valid metric, and $B(x, 1) = \{x\}$ for all $x \in X$. So every one-point subset of X is open in the metric topology, so the metric topology equals the discrete topology.

C. Theorem: Let $f : X_T \to Y_S$ be continuous and injective such that $f(X)$ is Hausdorff. Then X_T is Hausdorff.

Proof: Let $x \neq y \in X_T$. Because f is injective, $f(x) \neq f(y) \in f(X)$. Because $f(X)$ is Hausdorff, there exist disjoint neighborhoods U, V of $f(x)$, $f(y)$ respectively. Because f is continuous, $f^{-1}(U)$, $f^{-1}(V)$ are neighborhoods of x, y respectively. Because f is well-defined, $f^{-1}(U)$, $f^{-1}(V)$ are disjoint. So we conclude that X_T is Hausdorff.

[The original claim is indeed false. To see so, consider the case where Y_S is indiscrete. The new claim does indeed require injectivity. To see so, consider the case of a constant map.]

D.A. The flaw happens in the fourth sentence of the argument, where it is claimed that we have separations of Y_1 and Y_2. The flaw is that some of U_1, V_1, U_2, and V_2 may be empty.

D.B. No, the proposed theorem is incorrect. A simple counter-example is $X = \mathbb{R}$, $Y_1 = [0, 1]$, and $Y_2 = [2, 3]$.

D.C. The theorem becomes true if we add the hypothesis that $Y_1 \cap Y_2 \neq \emptyset$. It is then a special case of Munkres's Section 23 Exercise 2.