This exam is due on paper at the start of class on Wednesday. The exam must be completed within a single 180-minute (three-hour) block of time. Your block begins when you open (or peek inside) this packet.

The exam is open-note: You may use your class notes, your old homework, your textbook, and the course web site. You may not share any of these materials with other students. You may not discuss the exam with anyone but me until Wednesday at the start of class. You may not consult other Internet sites, artificial intelligences, etc.

I will try to check my e-mail frequently during the exam period. Feel free to ask clarifying questions. If you cannot obtain clarification on a problem, then explain your interpretation of the problem in your solution, so that I can judge your solution relative to your interpretation. You might lose points, if your interpretation makes a problem drastically easier than it should be. Certainly you should never interpret a problem in a way that renders it trivial.

Your solutions should be thorough, self-explanatory, neat, concise, and polished. Always show enough work and justification so that a typical classmate could understand your solutions. You may cite material (definitions, theorems, examples, etc.) from class, homework, assigned textbook readings, etc. You do not have to redevelop or reprove that material. If you wish to use material that we haven't studied, then you have to develop it. Correct answers without supporting work rarely earn full credit.

If you cannot solve a problem, then write a brief summary of what you know that is relevant, and the approaches that you've tried. Partial credit is often awarded.

Good luck. :)

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There are four problems labeled B–E, along with time checks A, F, which are not optional.

A. On what day, at what time, are you starting this exam?

For problem B, suppose that we have a covering map $p : E \to B$. Then we have a notion of lifting any map $f : X \to B$ to a map $\tilde{f} : X \to E$. So the natural (?!) thing to do is to study the special case where $X = E$ and $f = p$.

B. Faced with this situation, your friend conjectures that $\tilde{p}$ must be the identity map. Why does your friend think so? Are they right? Prove or disprove so.

For problem C, suppose that you are given topological spaces X and Y. Consider the set

$$X \sqcup Y = (\{0\} \times X) \cup (\{1\} \times Y).$$

Define a topology on $X \sqcup Y$ by declaring that subsets of the form $\{0\} \times U$ are open (for U open in X), and subsets of the form $\{1\} \times V$ are open (for V open in Y), and unions of those are open.

C.A. Let's do an example about intervals in $\mathbb{R}$. Is $(0, 2) \cup (3, 5)$ homeomorphic to $(0, 2) \sqcup (3, 5)$? Is $(0, 2) \cup (1, 3)$ homeomorphic to $(0, 2) \sqcup (1, 3)$? Explain, but rigorous proofs are not expected.
C.B. If X and Y are compact, then is $X \sqcup Y$ also compact? Prove or disprove so.
C.C. How does the fundamental group of $X \sqcup Y$ relate to those of X and Y? Explain.

The Borsuk-Ulam theorem for $\mathbb{S}^2$ (Theorem 57.3) is not part of this exam's material, but the Borsuk-Ulam theorem for $\mathbb{S}^1$ is fair game.

D. Use the Borsuk-Ulam theorem for $\mathbb{S}^1$ to invent a version of the Borsuk-Ulam theorem for the torus. Explain it. You might want to prove it, to check that it is correctly stated with all necessary assumptions, but the proof itself is not worth any points. (This is not intended to be a giant exercise where you invent bold new math. It's supposed to be a medium-sized exercise where you combine facts, that you already know, in a new way.)

For problem E, you should skim the definition of the compact-open topology on page 285.

E.A. Let X be a space, and fix $x \in X$. Describe how to endow $\pi_1(X, x)$ with a topology, so that it is a topological space. Be precise and thorough. (My solution has several steps.)
E.B. In the case of $X = \mathbb{S}^1$, where $\pi_1(X, x)$ is group-isomorphic to $\mathbb{Z}$, it is natural to wonder whether $\pi_1(X, x)$ is also homeomorphic to $\mathbb{Z}$ (under its standard topology). Unpack your construction from problem E.A as far as you can, so that you can state explicitly what must be checked. (Do not try to complete the check. It is too difficult for this exam.)

F. On what day, at what time, are you finishing this exam?

The following paragraph has nothing to do with the exam. It is placed here merely to take up space. You are not obligated to read it at all.

In middle school, my classmates and I were assigned to write essays about what it would be like to live in a utopia — a perfect place. We discussed the topic a bit, and I got the impression that everyone was going to write that living in a utopia would be boring. So my essay argued that it is logically impossible for a perfect place to be boring, because the boringness would be an imperfection. That's a powerful argument, but it doesn't really have any subtleties to explore, so my essay just kept saying it again and again in different words. The teacher and I eventually agreed that my argument was perfect but boring.