B. A lifting of $p : E \to B$ is a function $\tilde{p} : E \to E$ such that $p \circ \tilde{p} = p$. The identity map $i : E \to E$ certainly satisfies $p \circ i = p$ and thus is a lifting of p. But *must* a lifting of p be i?

If we could cancel p from both sides of the equation $p \circ \tilde{p} = p$, then we could conclude that $\tilde{p} = i$. We can do this when p is injective and thus bijective and thus invertible.

However, when p is not injective, $\tilde{p}$ is not necessarily the identity map, because liftings are not necessarily unique. Consider for example our most-studied covering map, where $E = \mathbb{R}$, $B = \mathbb{S}^1$, and $p(x) = (\cos 2\pi x, \sin 2\pi x)$. Then $\tilde{p}(x) = x + k$, for any integer k, satisfies $p \circ \tilde{p} = p$.

[In fact, this kind of map $\tilde{p}$ ends up being important to the theory of covering spaces. It is called a *deck transformation* or *covering transformation*. See Section 81.]

C.A. Yes, $(0, 2) \cup (3, 5)$ is homeomorphic to $(0, 2) \sqcup (3, 5)$ by the function

$$f(x) = \begin{cases} (0, x) & \text{if } x \in (0, 2), \\ (1, x) & \text{if } x \in (3, 5). \end{cases}$$

However, $(0, 2) \cup (1, 3)$ is not homeomorphic to $(0, 2) \sqcup (1, 3)$, because the former is a single interval $(0, 3)$ while the latter is homeomorphic to the union of two disjoint intervals.

[By the way, this $\sqcup$ operation on spaces is called the *disjoint union*.]

C.B. Yes, $X \sqcup Y$ must be compact. Before I prove so, notice that $U = \{0\} \times X$ and $V = \{1\} \times Y$ are open subsets of $X \sqcup Y$. Moreover, U is homeomorphic to X, and V is homeomorphic to Y.

Let $\{U_j\}_{j \in J}$ be an open cover of $X \sqcup Y$. Then $\{U \cap U_j\}_{j \in J}$ is an open cover of U, which is compact. So there exists a finite subset $K \subseteq J$ such that $\{U \cap U_j\}_{j \in K}$ is an open cover of U. Similarly, there exists a finite subset $L \subseteq J$ such that $\{V \cap U_j\}_{j \in L}$ is an open cover of V. Then $\{U_j\}_{j \in K \cup L}$ is a finite subcover of the original open cover of $X \sqcup Y$. So $X \sqcup Y$ is compact.

C.C. Well, $X \sqcup Y$ is not connected (unless X or Y is empty), because the U and V defined above are non-empty open subsets that partition $X \sqcup Y$. So $X \sqcup Y$ is not path connected.

So it seems that $\pi_1(X \sqcup Y, z)$ depends on where z is. If $z \in U$, then $z = (0, x)$ for some x, and $\pi_1(X \sqcup Y, z)$ is naturally isomorphic to $\pi_1(X, x)$. If $z \in V$, then $z = (1, y)$ for some y, and $\pi_1(X \sqcup Y, z)$ is naturally isomorphic to $\pi_1(Y, y)$.

D. [We know that any map $f : \mathbb{S}^1 \to \mathbb{R}^1$ must have a point $x \in \mathbb{S}^1 \subseteq \mathbb{R}^2$ such that $f(x) = f(-x)$. This fact was proved in our Day 09 homework. Because the torus is $\mathbb{S}^1 \times \mathbb{S}^1$, we should be able to say something similar about maps from it. I came up with Claim 0 below. Students came up with many other answers, a few of which are listed below.]

Claim 0: Let $f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R} \times \mathbb{R}$ be a map of the form $f(x, y) = (g(x), h(y))$. Then there exists a point $(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ such that $f(x, y) = f(-(x, y))$.

Proof 0: We know that $g : \mathbb{S}^1 \to \mathbb{R}$ has an x such that $g(x) = g(-x)$, and similarly there exists a y such that $h(y) = h(-y)$. Then

$$f(-(x,y)) = f(-x,-y) = (g(-x), h(-y)) = (g(x), h(y)) = f(x,y).$$

So the point (x, y) has the desired property.

False claim 1: Let $f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^2$ be a map. Then there exists a point $(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ such that $f(x, y) = f(-(x, y))$.

Disproof 1: Define f by $f(x, y) = x \in \mathbb{R}^2$. Then, for all $(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1$,

$$f(-(x,y)) = f(-x,-y) = -x \neq x = f(x,y).$$

False Claim 2: Embed $\mathbb{S}^1 \times \mathbb{S}^1$ into $\mathbb{R}^3$ so that it is symmetric about the origin. Then every map $f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^2$ has an x such that $f(x) = f(-x)$.

Disproof 2: Let f be the orthogonal projection onto the x-y-plane.

Claim 3: Let $f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$ be a map. Then along every meridian and every parallel there are antipodal points that have the same value under f. Precisely, for all $x \in \mathbb{S}^1$ there exists a $y \in \mathbb{S}^1$ such that $f(x, y) = f(x, -y)$, and for all $y \in \mathbb{S}^1$ there exists an $x \in \mathbb{S}^1$ such that $f(x, y) = f(-x, y)$.

Proof 3: The restriction of f to any meridian or parallel (or in fact any other homeomorphic copy of $\mathbb{S}^1$) obeys the one-dimensional Borsuk-Ulam theorem.

Claim 4: Let $f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$ be a map. Then there exists an $(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1$ such that $f(x, y) = f(-(x, y))$.

Proof 4: Define $g : \mathbb{S}^1 \to \mathbb{R}$ by $g(x) = f(x, x)$. Then, by the one-dimensional Borsuk-Ulam theorem, there exists an x such that $g(x) = g(-x)$. So $f(x, x) = f(-x, -x) = f(-(x, x))$.

Claim 5: Let $f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R} \times \mathbb{R}$ be a map. Then there exists a point $(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ such that $f(x, y) = (f_1(-x, y), f_2(x, -y))$.

Proof hint 5: Apply one-dimensional Borsuk-Ulam to $\mathbb{S}^1 \hookrightarrow \mathbb{S}^1 \times \mathbb{S}^1 \xrightarrow{f} \mathbb{R}^2 \xrightarrow{\pi_1} \mathbb{R}^1$. Then do the same for π_2. Then play them against each other.

E.A. Let X be a space and $x \in X$. Consider the set $\mathcal{C}(I, X)$ of all maps $f : I \to X$. Endow $\mathcal{C}(I, X)$ with the compact-open topology defined on page 285. Consider the subset $L \subseteq \mathcal{C}(I, X)$ consisting of maps f such that $f(0) = x = f(1)$. Endow L with the subspace topology. Via the natural surjection $L \to \pi_1(X, x)$, which sends a loop f to its path homotopy class $[f]$, endow $\pi_1(X, x)$ with the quotient topology.

E.B. The standard topology on $\mathbb{Z}$ is discrete. So we need to check, for all $[f] \in \pi_1(X, x)$, that the one-point set $\{[f]\} \subseteq \pi_1(X, x)$ is open. By the definition of the quotient topology, $\{[f]\}$ is open in $\pi_1(X, x)$ if and only if $[f] \subseteq L$ is open in L. By the definition of the subspace topology, $[f]$ is open in L if and only if there exists a U open in $\mathcal{C}(I, X)$ such that $[f] = L \cap U$. By the definition of the compact-open topology (and how a topology is generated by a subbasis), U is open in $\mathcal{C}(I, X)$ if and only if it is of the form

$$U = \bigcup_{j \in J} \bigcap_{k \in K_j} S(C_k, U_k),$$

where each K_j is finite. So in summary we must check, for all loops f based at x in X, that there exist $\{(C_k, U_k)\}_{k \in K_j, j \in J}$ such that a loop g is path homotopic to f if and only if $g \in \bigcup \bigcap S(C_k, U_k)$.