This exam should have five sheets of paper, including this cover page and problems A–D, printed single-sided. Feel free to use the backs of pages.

You have 2.5 hours (150 minutes) to complete this exam.

No notes, books, calculators, computers, etc. are allowed, except for your two-sided, one-sheet crib sheet that we have discussed.

Show all of your work, in as organized a manner as possible. Incorrect answers with solid work often earn partial credit. Correct answers without explanatory work rarely earn full credit. **Every question implicitly says, "Explain".**

Good luck. :)

A.A. For which integers k does there exist a compact, connected surface X such that $k = \chi(X)$?

A.B. For which integers k does there exist a compact surface X such that $k = \chi(X)$?

B. Using an argument about cutting and pasting collections of labeled polygonal regions, show that $\mathbb{RP}^2$ with a small open disk removed is homeomorphic to the (closed) Möbius strip.

Here are four compact, connected 3-manifolds: $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{S}^3$, $\mathbb{RP}^1 \times \mathbb{RP}^2$, $\mathbb{RP}^3$.

C. What are their fundamental groups? Based on those groups, which ones might be homeomorphic to each other? (Do not prove any homeomorphism.)

Consider the torus $\mathbb{T}^2$ embedded in $\mathbb{R}^3$. Let N be a small neighborhood of $\mathbb{T}^2$, such as

$$N = \{\vec{x} \in \mathbb{R}^3 : \exists \vec{y} \in \mathbb{T}^2 \; d(\vec{x}, \vec{y}) < \epsilon\}$$

for some small $\epsilon > 0$. Let U_1 be the union of N and all of the points of $\mathbb{R}^3$ that are enclosed by ("inside of") $\mathbb{T}^2$. So U_1 is an open solid torus.

This entire situation embeds into $\mathbb{S}^3$ via the inverse of stereographic projection. Let U_2 be the union of N and all of the points of $\mathbb{S}^3$ that are not enclosed by ("outside of") $\mathbb{T}^2$. So $\mathbb{S}^3 = U_1 \cup U_2$. It turns out that U_2 is an open solid torus homeomorphic to U_1. (Assume so.)

D. What does the Seifert–van Kampen theorem say about this situation? (Be detailed. Several groups are involved. If you know their isomorphism type, then say so. If you know how they relate, then say so. But you are not supposed to do an intense group theory proof.)