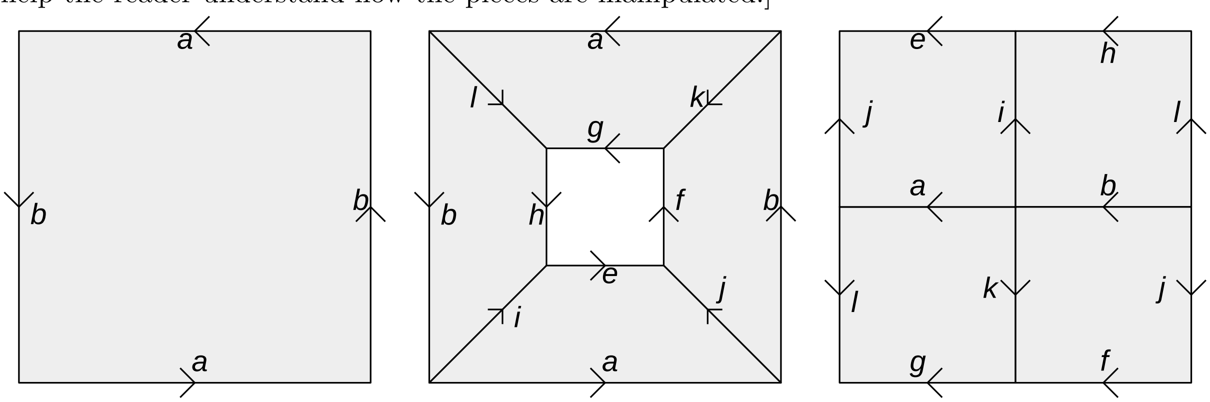
A.A. The short answer is: all integers $k \leq 2$. From class we know that $\chi(\mathbb{S}^2) = 2$ and $\chi(\mathbb{RP}^2) = 1$. From homework we know that $\chi(X \# Y) = \chi(X) + \chi(Y) - 2$. So the Euler characteristic of the connected sum of g real projective planes is $2 - g$, and we get $k = 2, 1, 0, -1, \ldots$. Incorporating tori does not give us any other k, because the connected sum of g tori has Euler characteristic $2 - 2g$. There is no way to get $k \geq 3$ until problem A.B.

A.B. The short answer is: all integers k. From homework we know that $\chi(X \sqcup Y) = \chi(X) + \chi(Y)$. Therefore the disjoint union of $k \geq 3$ copies of $\mathbb{RP}^2$ has Euler characteristic k. This construction handles all k not already handled in problem A.A.

B. Below left is $\mathbb{RP}^2$. Below center is $\mathbb{RP}^2$ with an open disk removed. I have added four proto-cuts labeled i, j, k, l, in preparation for the next step. We cut along j and l and glue along a and b to obtain the labeled polygon below right. That labeled polygon is the closed Möbius strip, because the left and right sides glue with a half twist and the top and bottom sides remain unglued. [The i and k proto-cuts are never actually used. I include them because they might help the reader understand how the pieces are manipulated.]



C. First,

$$\pi_1(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1, x) \cong \mathbb{Z}^3$$

by a covering space argument or the product theorem for fundamental groups. Second, we know from class that $\pi_1(\mathbb{S}^3, x)$ is trivial. Third,

$$\pi_1(\mathbb{RP}^1 \times \mathbb{RP}^2, (x, y)) \cong \pi_1(\mathbb{RP}^1, x) \times \pi_1(\mathbb{RP}^2, y) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

by the product theorem, the fact that $\mathbb{RP}^1$ is homeomorphic to $\mathbb{S}^1$, and what we know of $\mathbb{RP}^2$ from class. Finally, we know from class that $\pi_1(\mathbb{RP}^3, x) \cong \mathbb{Z}/2\mathbb{Z}$. These four groups are distinct (not isomorphic to each other), so these four manifolds must be distinct (not homeomorphic to each other).

D. The assumptions of the Seifert–van Kampen theorem are met: U_1 and U_2 are open subsets of $\mathbb{S}^3$, and each of them is path connected, and $U_1 \cap U_2 = N$ is path connected. Let $x \in N$.

Seifert–van Kampen says that the inclusions $U_k \hookrightarrow \mathbb{S}^3$ induce an isomorphism

$$\frac{\pi_1(U_1, x) * \pi_1(U_2, x)}{K} \to \pi_1(\mathbb{S}^3, x),$$

where K is the smallest normal subgroup containing all words of the form $(i_{1*}(c)^{-1}, i_{2*}(c))$ for $c \in \pi_1(N, x)$, and $i_k : N \hookrightarrow U_k$ is the inclusion.

We know that $\pi_1(U_1, x) \cong \mathbb{Z} \cong \pi_1(U_2, x)$ (by deformation retracting a solid torus onto a circle), and $\pi_1(N, x) \cong \mathbb{Z} \times \mathbb{Z}$ (by deformation retracting N onto a torus). We even know $\pi_1(\mathbb{S}^3, x) \cong \{0\}$ (from class). So K must be the entire group $\pi_1(U_1, x) * \pi_1(U_2, x)$.

Here's more insight into K. Let a_1 and a_2 be generators of $\pi_1(U_1, x)$ and $\pi_1(U_2, x)$ respectively. Then $a_1 = i_{1*}(b_1)$ and $a_2 = i_{2*}(b_2)$, where b_1 and b_2 are "parallel" and "meridian" generators of $\pi_1(N, x)$. So K contains both

$$a_1 \sim (a_1, 1) = (i_{1*}(b_1^{-1})^{-1}, i_{2*}(1))$$

and

$$a_2 \sim (1, a_2) = (i_{1*}(1), i_{2*}(b_2)).$$

That explains, at least sketchily, how K ends up being all of $\pi_1(U_1, x) * \pi_1(U_2, x)$.

[By the way, here is my rubric, listing how this problem's 12 points were allocated:

1: assumptions are met

2: induced surjection from free product

2: $\pi_1(U_1, x) \cong \mathbb{Z} \cong \pi_1(U_2, x)$

2: $\pi_1(N, x) \cong \mathbb{Z} \times \mathbb{Z}$

2: $\pi_1(\mathbb{S}^3, x) \cong \{0\}$

2: so the kernel is the whole group

1: insight into why the kernel should be the whole group.]