1 Review

Suppose that we're working in n dimensions. Formally, let

$$\nabla = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right\rangle.$$

"Formally" means "manipulating symbols according to familiar rules, without worrying about whether they mean anything". We have to say "formally" because ∇ looks like a vector but it's not a vector; it's some other weird kind of mathematical object. Anyway, formally, for any scalar field f,

$$\begin{aligned} \nabla f &= \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right\rangle f \\ &= \left\langle \frac{\partial}{\partial x_1} f, \frac{\partial}{\partial x_2} f, \ldots, \frac{\partial}{\partial x_n} f \right\rangle, \end{aligned}$$

which agrees with our usual notation for ∇f as the gradient of f. Similarly, for any vector field $\vec{F}$,

$$\begin{aligned} \nabla \cdot \vec{F} &= \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right\rangle \cdot \langle F_1, F_2, \ldots, F_n \rangle \\ &= \frac{\partial}{\partial x_1} F_1 + \frac{\partial}{\partial x_2} F_2 + \ldots + \frac{\partial}{\partial x_n} F_n, \end{aligned}$$

which is the divergence of $\vec{F}$. And it's reasonable to ask, in the special case of $n = 3$, what happens when we formally compute the cross product of ∇ with a vector field $\vec{F}$:

$$\begin{aligned} \nabla \times \vec{F} &= \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right\rangle \times \langle F_1, F_2, F_3 \rangle \\ &= \left\langle \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right\rangle. \end{aligned}$$

This vector field is the curl of $\vec{F}$. To summarize, in $\mathbb{R}^3$ we have

$$\begin{aligned} \text{grad } f &= \nabla f, \\ \text{curl } \vec{F} &= \nabla \times \vec{F}, \\ \text{div } \vec{F} &= \nabla \cdot \vec{F}. \end{aligned}$$

Each of these concepts has a geometric meaning. The gradient ∇f tells you, at each $\vec{x}$, how to climb the hill defined by f as quickly as possible there. The curl $\nabla \times \vec{F}$ tells you, at each $\vec{x}$, how much $\vec{F}$ rotates or curls there. The divergence $\nabla \cdot \vec{F}$ tells you, at each $\vec{x}$, whether $\vec{F}$ spreads out or comes together there. See our Math 211 Mathematica notebooks for many examples.

These concepts also have many applications. For example, almost everything that we encounter in daily life is a consequence of electromagnetism, which (at least in its pre-quantum theory) is governed by Maxwell's equations. In the simplest conditions, these equations relate a time-dependent vector field $\vec{E}$ for electricity and a time-dependent vector field $\vec{H}$ for magnetism as follows.

$$\text{div } \vec{E} = 0, \quad \text{div } \vec{H} = 0, \quad \text{curl } \vec{E} = -\frac{\partial}{\partial t}\vec{H}, \quad \text{curl } \vec{H} = \frac{\partial}{\partial t}\vec{E}.$$

For another application, suppose that you're studying how water flows (through a pipe, along the hull of a boat, etc.). The flow is quantified as a velocity field $\vec{v} = \vec{v}(t, \vec{x})$ and a pressure field $p = p(t, \vec{x})$, which (under reasonable assumptions, and for certain constants δ, ν) are governed by the Navier-Stokes equations

$$\begin{aligned}
\frac{\partial v_1}{\partial t} + \vec{v} \cdot \nabla v_1 &= -\frac{1}{\delta}\nabla p + \nu \nabla \cdot \nabla v_1, \\
\frac{\partial v_2}{\partial t} + \vec{v} \cdot \nabla v_2 &= -\frac{1}{\delta}\nabla p + \nu \nabla \cdot \nabla v_2, \\
\frac{\partial v_3}{\partial t} + \vec{v} \cdot \nabla v_3 &= -\frac{1}{\delta}\nabla p + \nu \nabla \cdot \nabla v_3, \\
\nabla \cdot \vec{v} &= 0.
\end{aligned}$$

So I hope that you are starting to believe me, that grad, curl, and div are important in science and engineering.

2 Stokes's theorem

The table below summarizes five big theorems of calculus, three of which we have studied.

dimension	flat version	curved version
1	$\int_{[a,b]} F'(x_1)\, dx_1 = F(b) + -F(a).$	$\int_C \nabla f \cdot d\vec{x} = f(\vec{x}(b)) + -f(\vec{x}(a)).$
2	$\iint_D \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\, dx_1\, dx_2 = \int_C \vec{F} \cdot d\vec{x}.$	$\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{x}.$
3	$\iiint_E \text{div } \vec{G}\, dx_1\, dx_2\, dx_3 = \iint_S \vec{G} \cdot d\vec{S}.$	

The upper-left theorem is the fundamental theorem of calculus. The upper-right theorem is the fundamental theorem of calculus for line integrals. Both the interval $[a, b]$ and the curve C are one-dimensional. The interval $[a, b]$ is essentially a curve C that happens to be "flat".

The middle row of the table features Green's theorem on the left and Stokes's theorem (Section 16.8) on the right. Both the planar region D and the surface S are two-dimensional. The planar region D is essentially a surface S that happens to be flat.

The third row of the table features the divergence theorem (Section 16.9) on the left. It concerns an integral over a solid three-dimensional region E, and another integral over the surface S that bounds E. The right side of the third row is left blank, because the corresponding theorem over three-dimensional "curved solids" is rarely taught in calculus courses.

All five theorems listed in the table follow a single pattern. In each equation, the right integrand is a function, and the left integrand is some kind of derivative of that function. In each equation, the left integral is over an n-dimensional space, and the right integral is over its $(n-1)$-dimensional boundary. (The boundary of a one-dimensional space is a zero-dimensional space, which is a set of discrete points. Those points are assigned positive and negative signs according to a certain rule. An integral over a set of discrete points is simply a sum.)

The table doesn't end at three dimensions. There is a way to systematize everything that we've learned, so that it extends to curved or flat spaces of all dimensions. The integrand and the $dx_1\ dx_2\ \ldots$ get combined into an object called a *differential form*, denoted something like ω. A general derivative of this ω, denoted $d\omega$, can be rigorously defined. The boundary of a space X is denoted ∂X. Then, under appropriate conditions, we obtain a single, grand theorem that works in all dimensions:

$$\int_X d\omega = \int_{\partial X} \omega.$$

It's called Stokes's theorem, in honor of the original Stokes's theorem.

We can't possibly learn all of this stuff in Math 211. At Carleton College, sometimes it's learned in special topics courses such as Math 295 or Math 395. Sometimes Comps projects are done on this topic. Most Math majors graduate without learning it. I first learned about it in graduate school. So why do I mention it in Math 211? To emphasize that there is a pattern to the five theorems. To give you a hint that, if you go on in mathematics, you will find the subject getting more abstract but, surprisingly, simpler.

3 de Rham cohomology

Let's return to three dimensions specifically. Let $\mathcal{S}$ be the set of all scalar fields $f : \mathbb{R}^3 \to \mathbb{R}$ that are smooth — meaning, every derivative that you can imagine, such as $f_{x_1x_3x_3x_2x_1x_1x_1x_2x_2}$, is continuous. Let $\mathcal{V}$ be the set of all vector fields $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ that are smooth in each of the three components. Then the gradient, curl, and divergence fit into this diagram of functions:

$$\mathcal{S} \xrightarrow{\text{grad}} \mathcal{V} \xrightarrow{\text{curl}} \mathcal{V} \xrightarrow{\text{div}} \mathcal{S}.$$

In other words, the gradient of a smooth scalar field is a smooth vector field, the curl of a smooth vector field is a smooth vector field, and the divergence of a smooth vector field is a smooth scalar field.

Now here's an important fact: For any vector field $\vec{F}$, $\text{div}(\text{curl}\ \vec{F}) = 0$. In other words, if $\text{div}\ \vec{G} \neq 0$, then $\vec{G}$ is not the curl of any vector field $\vec{F}$.

Exercise A: Prove the claim just made, that $\text{div}(\text{curl}\ \vec{F}) = 0$ everywhere, for any $\vec{F}$.

What about the converse? Are there any divergence-0 vector fields $\vec{G}$ that are not curls? Surprisingly, the answer depends on whether there are any holes in the domain of $\vec{G}$. The crucial example is

$$\vec{G} = \frac{\vec{x}}{|\vec{x}|^{3/2}}$$

(or anything proportional to it, such as the Newtonian gravitational force field). It is undefined at the point $\vec{0}$. Everywhere else, its divergence is 0. However, there is no $\vec{F}$, defined everywhere that $\vec{G}$ is defined, such that curl $\vec{F} = \vec{G}$.

Exercise B: Here's a sketch of the proof that the $\vec{G}$ just defined cannot be a curl. Consider the flux of $\vec{G}$ across the unit sphere. Intuitively, do you think that it's positive, negative, or zero? Then, assuming that $\vec{G} = \text{curl } \vec{F}$, this flux is the left side of the original Stokes's theorem. So what does Stokes's theorem say about it?

Here's another important fact: For any scalar field f, curl(grad f) $= \vec{0}$ everywhere. In other words, if curl $\vec{F} \neq \vec{0}$, then $\vec{F}$ is not the gradient of any scalar field f.

Exercise C: Prove the claim just made, that curl(grad f) $= \vec{0}$ everywhere, for any f.

What about the converse? Are there any curl-$\vec{0}$ vector fields $\vec{F}$ that are not gradients? Again, the answer depends on holes in the domain of $\vec{F}$. The crucial example is

$$\vec{F} = \frac{\langle -x_2, x_1, 0 \rangle}{x_1^2 + x_2^2}.$$

This vector field is undefined along the line $x_1 = x_2 = 0$. Everywhere else, its curl is $\vec{0}$. However, there is no f, defined everywhere that $\vec{F}$ is defined, such that grad $f = \vec{F}$.

Exercise D: Prove the claim just made, that this specific $\vec{F}$ cannot have a potential function. (Hint: If it did, then that would imply the existence of a potential function for a certain two-dimensional vector field, and we already know that that potential function cannot exist.)

Thus far, we've increased our understanding of $\vec{G}$ with divergence 0 and $\vec{F}$ with curl $\vec{0}$. Maybe we should study f with gradient $\vec{0}$ too? If grad $f = \vec{0}$ everywhere, then f must be constant, right? Actually, the answer depends on holes in the domain of f.

Exercise E: By allowing yourself to place holes in the domain of f, find a non-constant f such that grad $f = \vec{0}$ everywhere f is defined. (Hint: So far we've seen point-shaped holes and line-shaped holes. What's next in that sequence?)

At the other end of the diagram, there should be something about scalar fields that aren't divergences, right? But there's a twist:

Exercise F: Prove that every scalar field f is the divergence of some vector field $\vec{F}$.

In summary, if we are studying a subset of $\mathbb{R}^3$, then the existence of scalar fields and vector fields with certain properties on that subset is intimately connected to whether that subset has holes, and how those holes are shaped (and how many there are). In other words, there is an interplay between delicate, finicky questions of calculus and simple, crude questions of topology (the study of how spaces connect up on themselves). In higher mathematics, these ideas get systematized in a concept called de Rham cohomology, which like the general Stokes's theorem is quite lovely.