For the first half of this material, you can consult almost any analysis textbook. For the second half, a good reference is Section 57 of these functional analysis lecture notes by Sigurd Angenent:

`https://people.math.wisc.edu/~angenent/Free-Lecture-Notes/725notes.pdf`

Definition 9.1.1. *Let (X, d) be a metric space. A sequence $x_0, x_1, x_2, \dots$ of points in X is said to be* ***Cauchy*** *if for all $\epsilon > 0$ there exists an N such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$. Moreover, (X, d) is* ***complete*** *if every Cauchy sequence converges.*

Example 9.1.2. *Let $T > 0$. Let $C([0, T], \mathbb{R})$ be the set of all continuous functions $x : [0, T] \to \mathbb{R}$. It is an infinite-dimensional vector space. On it, define the* sup *norm*

$$\|x\| = \sup_{t \in [0,T]} |x(t)| = \max_{t \in [0,T]} |x(t)|.$$

Then $C([0, T], \mathbb{R})$ is a complete metric space under the metric induced by this norm.

Definition 9.1.3. *A function $f : (X, d_X) \to (Y, d_Y)$ is* ***Lipschitz continuous*** *if there exists a constant $K \geq 0$ such that $d_Y(f(x), f(y)) \leq K d_X(x, y)$. Moreover, a Lipschitz continuous $F : (X, d) \to (X, d)$ with $K < 1$ is a* ***contraction****.*

The intuition that you should have is that Lipschitz continuity is stronger than continuity but weaker than continuous differentiability (C^1). My sense is that analysts like having these fine gradations of continuity, because they illuminate exactly what is required to prove theorems, while topologists consider the fine gradations to be too fussy.

Theorem 9.1.4. *A contraction of a complete metric space has a unique fixed point. That is, if (X, d) is complete and $F : (X, d) \to (X, d)$ is a contraction, then there exists a unique $x \in X$ such that $F(x) = x$.*

Proof. Here's a sketch of the key steps. Starting from any $x_0 \in X$, define $x_{n+1} = F(x_n)$. You get a Cauchy sequence, which converges to some $x \in X$. That's the fixed point. Moreover, it's unique. □

Theorem 9.1.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous with constant L. Let $0 < T < L$. Then the ordinary differential equation $\frac{dx}{dt} = f(x)$ with initial condition $x(0) = x_0$ has a unique solution $x \in C([0, T], \mathbb{R})$.*

Proof. By the fundamental theorem of calculus, the differential equation is tantamount to the integral equation

$$x(t) = x_0 + \int_0^t f(x(s))\, ds.$$

Define $F : C([0, T], \mathbb{R}) \to C([0, T], \mathbb{R})$ by

$$F(x)(t) = x_0 + \int_0^t f(x(s))\, ds.$$

Then $x \in C([0, T], \mathbb{R})$ solves the integral equation if and only if $F(x) = x$. Well, it can be shown that F is a contraction on $C([0, T], \mathbb{R})$ with $K = LT$. So it has a unique fixed point x. □